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# ON THE HISTORY OF THE DEVELOPMENT OF THE NONHOLONOMIC DYNAMICS

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It is possible to single out two directions in the development of the nonholonomic dynamics, and in each of these directions one can find interesting investigations. One of these investigations is connected with the general formalism of the equations of dynamics that differs from the Lagrangian and Hamiltonian method of the equations of motion's construction. Historically, certain errors of the well-known mathematicians, among which we can name C. Neumann [33] and E. Lindelöf [32], were caused by an incorrect application of the Lagrange equations in the presence of nonintegrable constraints in the description of the problem of a body rolling without sliding on the horizontal plane. The general understanding of inapplicability of Lagrange equations and variational principles to the nonholonomic mechanics is due to H. Herz, who expressed it in his fundamental work *Die Prinzipien der Mechanik in neuem Zusammenhange dargestellt* [4] that deals mostly with his conception of hidden cyclic parameters (coordinates, masses), as opposed to the conventional notion of interaction as a result of force application.

Herz's observations were developed by H. Poincaré [13] in his well-known paper *Herz's Ideas in Mechanics*. From this paper, we would like to quote some paragraphs, which illustrate, though in somewhat epistolary form, the views of both scholars.

"Herz terms a system *holonomic* when the following holds: if the system's constraints do not allow a direct transition from one position to another infinitely close position, then they either do not allow indirect transitions between these positions. Only rigid constraints exist in such systems.

It is evident that our sphere is not a holonomic system.

So, it sometimes happens that the principle of least action cannot be applied to nonholonomic systems. Indeed, one can proceed from position $A$ to position $B$ taking the path that we have just discussed, or, undoubtedly, one of many other paths. Among these, there is, evidently, one path corresponding to the least action. Hence, it should have been possible for the sphere to follow this path from $A$ to $B$. But this is not so: whatever the initial conditions of motion may be, the sphere will never pass from $A$ to $B$.

In fact, if the sphere does pass from position $A$ to position $A'$, it does not always follow the path that corresponds to the least action.

The principle of least action holds no more.

Herz says, "In this case, a sphere obeying this principle would seem to be a living creature, which deliberately pursues a certain goal, while a sphere following the laws of Nature would look as an inanimate monotonously rolling mass... But such constraints do not exist in Nature. So-called *rolling without sliding* is actually rolling with slight sliding. This phenomenon belongs to the class of irreversible phenomena, such as

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friction; these phenomena are still poorly investigated, and we have not yet learned to apply to them the true principles of Mechanics."

Our reply is, "Rolling without sliding does not contradict either the law of energy conservation, or any other law of physics known to us. This phenomenon can be realized in the observable world within the accuracy that would allow its application to construction of the most accurate integration machines (planimeters, harmonic analyzers, etc.). We have no right to exclude it from consideration as impossible. As for our problems, they still remain regardless of whether such rolling is realized exactly or only approximately. To accept the principle, it is necessary to require that its application to a problem with almost exact source data would yield the results, as close to the exactness as the source data were. Besides, other (rigid) constraints can also be realized in Nature only approximately. But nobody is going to exclude them from consideration..."

The basic difference between nonholonomic dynamics and common Lagrangian one lies in the fact that the equations of constraints, written in terms of generalized coordinates $q_j$ and generalized velocities $\dot{q}_j$ in the following form:

$$f_i(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = 0, \qquad i = 1, \ldots, k, \quad \boldsymbol{q} = (q_1, \ldots, q_n), \tag{1}$$

cannot be presented in the final (integral) form

$$F_i(\boldsymbol{q}, t) = 0 \tag{2}$$

that sets limits only to the generalized coordinates. In this sense, one says that the constraints are nonintegrable (differential). According to Herz [4], they also can be called nonholonomic[1].

Historically, Ferrers's equations with undetermined multipliers $\lambda_1, \ldots, \lambda_k$ (1871)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i + \sum_j \lambda_j \frac{\partial f_j}{\partial q_i}. \tag{3}$$

should be considered as the first general form for the equations of nonholonomic mechanics. In equations (3), $T$ is the kinetic energy, $Q_i$ stand for the generalized forces, while $\lambda_j$ are undetermined multipliers, which, generally speaking, can be unambiguously recovered from the constraint condition $f(q, \dot{q}) = 0$. As a rule, the constraints studied in nonholonomic mechanics are linear with respect to the generalized velocities, i. e. realizable in conceptual problems,

$$f_i(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \sum_k a_{ik}(\boldsymbol{q}, t)\dot{q}_k + a_i(\boldsymbol{q}, t) = 0. \tag{4}$$

However, Bolzmann and Hamel refer to a somewhat artificial example of a nonlinear nonholonomic constraint. Ferrers also excluded the undetermined multipliers and obtained a sort of an analog of Lagrangian equations of motion [27].

Besides Ferrers's equations, nonholonomic dynamics also makes use of the equations that are due to Appell, Chaplygin, Magri, Volterra, and Bolzmann–Hamel. Such a diversity, generally speaking, can hardly be regarded as a major achievement. All these forms result from various ways of excluding the undetermined multipliers, and usually are not of much practical use. When setting up the specific equations to describe, for example, the process of rolling, one would usually use the general dynamic equations or the original equations in form (3).

The second direction substantially more important, in our opinion, for dynamics in general, this direction includes investigations concerning the analysis of the specific nonholonomic problems. The early statements of such problems date back to E. Routh, S. A. Chaplygin, P. V. Woronetz, P. Appell, and G. K. Suslov; these people have found remarkable integrable situations and provided them with proper analytical and qualitative descriptions. The majority of these problems also deal with rolling bodies. Together with searching for integrable cases, many researches were concerned with the stability

[1]The term *holonomic* was coined from two Greek roots ὅλος (whole, integrable) and νομος (law).

of particular solutions (permanent rotations, for example) of the general, nonintegrable, systems. Most widely known are, for example, the investigations concerning the stability of rotations of so-called *Woblestone* about its vertical axis; this stone shows a surprising dependency of its stability on the direction of its rotation. The most complete analytical results for this phenomenon were obtained by A. V. Karapetyan, I. S. Astapov, A. P. Markeev, and M. Pascal, but their work nowise exhausted the problem of description of the evolution of this system (some preliminary numerical results can be found in [31]). Certain properties of Woblestone with its distinctive noncoincidence between the geometrical and dynamical axes are still waiting for the proper theoretical explanation.

In the recent two decades, the development of the nonholonomic systems' studies has been associated with finding of new integrable problems by V. V. Kozlov, A. P. Markeev, A. P. Veselov, L. E. Veselova, Yu. N. Fedorov, A. V. Borisov, I. S. Mamaev, as well as with computer-aided and qualitative investigations of integrable and nonintegrable situations. In this connection, one should mention papers [26, 28, 29, 30, 34]. The majority of the said results open new horizons in the study of nonholonomic systems.

On the other hand, we would like to draw reader's attention to some recent papers dealing with the problems of nonholonomic reduction and almost Hamiltonian formulation of the equations of nonholonomic mechanics [21, 22, 23, 25]. These papers develop the method of reduction with the help of the symmetry, which, even for the case of Hamiltonian situation, has been over formalized by Marsden and Weinstein, rendering even the simplest facts dating back to Jacobi and Poisson completely unevident. Again, this formalism does not help to solve any new problem, neither its development in the nonholonomic case gives any conceptual results. As for the formulation of the equations of motion in the almost Hamiltonian form [35], for which Jacobi identity does not hold, one can only assert that such formulation in itself has only formal meaning, though certain dynamical effects exist that prevent the equations of nonholonomic mechanics from being formulated in the "true" Hamiltonian form. One of these effects, associated with the nonexistence of the invariant measure and with asymptotic properties, was mentioned by V. V. Kozlov in his fundamental work [10].

Unfortunately, the papers by S. A. Chaplygin [16, 17, 18], G. K. Suslov [15], Wagner [2], V. V. Kozlov were never translated into English, thus remaining unknown to the majority of the world's scientific community. The publication of the present and the next volumes is meant to partially fill the vacuum.

Let us also mention comparatively new and quite remarkable studies by V. A. Yaroshchuk [19, 20], who found new cases of the invariant measure's existence, as well as papers by A. V. Karapetyan [7] and V. V. Kozlov [9], concerning the question of realization of nonholonomic constraints. These papers develop the earlier studies by C. Caratheodori [24], who associated the question of the origin of nonintegrable constraints with infinitely large viscosity factor. To the readers interested in other models of the dynamics of systems with nonintegrable constraints, Dirac mechanics and vaconomic mechanics, we recommend reviews [1, 36].

The modern achievements in the studies of stability of nonholonomic systems are described in [8], more elementary issues are discussed in [12, 14].

There are only two special monographs dealing with nonholonomic mechanics: by Yu. I. Neimark, N. A. Fufaev [12] and by V. V. Dobronravov [6]. Both are quite out-of-date, and the latter contains a number of incorrect assertions. In newer reviews by P. Griffiths [5], A. M. Vershik and V. Ya. Gershkovich [3], the problems of nonholonomic mechanics are intermixed with those of nonholonomic geometry (the latter discipline was initiated by Cartan, Wagner and Rashevsky), and this intermixing has not yet brought any results for the problems of dynamics itself. In the said papers, for example, the equations of motion are postulated from the variational principle, invalidity of which for nonholonomic mechanics was stated as far back as by H. Herz. In this case, one obtains for a disk rolling on a plane the equations of vaconomic mechanics that do not describe rolling of bodies. (In vaconomic mechanics, the constraints are realized with the help of added masses; these and similar issues are discussed in papers [1, 9].) From the series of papers in question, only Wagner's paper [2] can,

apparently, be of any interest to a student of dynamics (in this paper, Wagner gave purely mechanical interpretation of the constraint of Suslov's problem in contradistinction of the erroneous realization of the same constraint given by Suslov himself [15]).

Considerably more notable and modern work by A. P. Markeev [11] stands out against the background of the mentioned monographs and reviews. It was published in 1992, but, unfortunately, is available only in Russian. Into the next volume we included two of the most interesting Markeev's papers from the mentioned book, one on the dynamics of a Woblestone and other on the integrable motion of a ball with a rotor.

To conclude, let us note that many historical details and new developments, not included in this essay, are dwelled upon in two our papers dealing with a rigid body rolling on a fixed surface without sliding, which is the classical field of application for the results of nonholonomic dynamics.

We also believe that these two papers are opening a new page in the study of nonholonomic systems, which is closely connected with the efficient use of computer-aided methods (systems of analytical calculations, visualization of motion, numerical experiments). Following this path, one can obtain results undreamed of in the times of analytical methods alone. It is in this new direction of research that the major discoveries of the decades to come will be made.

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